PROGRAM SYSTEM “DECISION-MAKING: PROCEDURES ON GRAPHS”

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Introduction

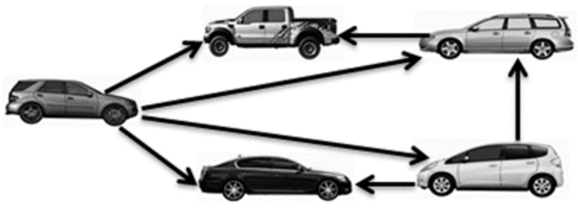
The study of the discipline "Mathematical Theory of Decision Making" allows you to solve a large number of practical problems of choosing the best options from the considered alternatives. Decision-making tasks arise in all areas of human activity, both in everyday life and in solving complex scientific and technical problems. Complex decisions have to be made at all stages of the creation of technology, starting with the development and selection of the best project.

The software system can also be used when studying the section "graph Theory" in the discipline "Discrete mathematics". This will demonstrate the practical value of procedures on graphs.

Object of the research

1. A software system for supporting the decision-making process developed.
2. Original algorithms for consistent aggregation of preferences and procedures for choosing and ordering alternatives by digraphs proposed.

Example. The decision-maker asked each family member to compare cars in pairs. Having learned the preferences of all relatives, he built an aggregate preference from the conditions of a simple majority in preferences, on the basis of which he chose the best option for the car.



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AGGREGATION OF INITIAL PREFERENCES ON GRAPHS

Formulation of the problem

Let the profile of expert preferences be given by the binary relations $\rho_1, \rho_2, \ldots, \rho_m$ on the set of alternatives $A = \{a_1,\ a_2, \ldots, a_n\}$.

We need to order alternatives by preference or choose the best.

First we need to build an aggregate preference relation.

Basic definition

To each binary relation ρ we associate a digraph $G =< A,\ \rho >$. Let the relations $\rho_1, \rho_2, \ldots, \rho_m$ be given by the adjacency matrices $R^1, R^2, \ldots, R^m$ of the corresponding graphs.

Definition 1. The digraph $G =< A,\ \rho_\Sigma >$ corresponding to the aggregated relation ρ_Σ constructed by the majority rule will be called the majority graph.

Definition 2. The distance between the relations ρ_k and ρ_t is called the quantity $d(\rho_k,\ \rho_t)$, determined by the formula

$$d(\rho_k, \rho_t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| r_{ij}^{k} - r_{ij}^{t} \right|. \quad (1)$$

The majority graph $G =< A,\ \rho_\Sigma >$ satisfies this condition

$$D(\rho_\Sigma) = \sum_{t=1}^{m} d(\rho_\Sigma, \rho_t) \to \min.$$

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The preference matrix of the relation ρ is the square (n × n) matrix $P = \|p_{ij}\|$ with elements

$$p_{ij} = \begin{cases} 1, if\, a_i \; is \; less \; preferred \; a_j; \\ \frac{1}{2}, if \; a_i and \; a_j \; are \; equivalen \;\; t; \\ 0, if \; a_j \; is \; less \; preferred \; a_i \; or \; a_i, a_j \; are \; not \; comparable. \end{cases}$$

Definition 3. The unstrict weighted majority graph is a weighted digraph $G =< A, \; \rho_\Sigma >$ with a set of vertex-alternatives $A = \{a_1, \; a_2, \dots, a_n\}$ and arcs $\rho_\Sigma = \{< a_i, a_j > | \; a_i, a_j \in A$ and $l_{ij} \geq 0\}$, where $l_{ij} = \sum_{k=1}^{m}(p_{ij}^k - p_{ji}^k)$. And each arc $< a_i, a_j > \in \rho_\Sigma$ put in accordance with the weight l_{ij}.

In this definition p_{ij}^k and p_{ji}^k are elements of the preference matrix P^k of the relation ρ_k $(i, j = 1, \dots, n, k = 1, \dots, m)$.

We define the weight (n × n) matrix of the majority graph $C = \|c_{ij}\|$:

$$c_{ij} = \begin{cases} l_{ij}, \text{if the arc} < a_i, a_j > \in \rho_\Sigma \,, \\ \infty, otherwise. \end{cases}$$

The matrix of total preferences $P = \|p_{ij}\|$, where $p_{ij} = \sum_{k=1}^{m} p_{ij}^k$.

$$c_{ij} = \begin{cases} p_{ij} - p_{ji}, \text{if the arc} < a_i, a_j > \in \rho_\Sigma \,, \\ \infty, otherwise. \end{cases} \quad (2)$$

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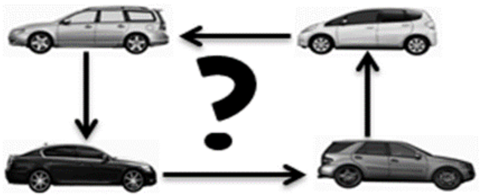
The relation ρ_t (t=1, …, n) is divided into symmetric $Sym\ \rho_t$ and asymmetric $As\ \rho_t$ parts. $Sym\ \rho_t$ includes all such pairs for which $< a_i, a_j > \in \rho_t$ and $< a_j, a_i > \in \rho_t$ simultaneously ($a_i, a_j \in A$). In this case, we assume that the alternatives a_i and a_j are equivalent. If $< a_i, a_j > \in \rho_t$, and $< a_j, a_i > \notin \rho_t$, then the alternative a_i is less preferable than a_j, and the pair $< a_i, a_j > \in As\rho_t$.



Definition 4. A cycle of the relation ρ is called contradictory if it has at least one pair $< a_i,\ a_j > \in As\rho$, ($a_i,\ a_j \in A$).

Definition 5. A relation ρ is called contradictory if it contains a contradictory cycle.

An algorithm for constructing a relation ρ that does not contain contradictory cycles by ρ_Σ

1. We check the digraph $G =< A,\ \rho_\Sigma >$ for the presence of contradictory cycles. If there are no such cycles, then the digraph $G =< A,\ \rho_\Sigma >$ without weights on the arcs is the desired digraph $G =< A,\ \rho_\Sigma >$ and $\rho_\Sigma = \rho$. If there are contradictory cycles, go to step 2.
2. From the digraph $G =< A,\ \rho_\Sigma >$ we remove all arcs $< a_i,\ a_j > \in As\rho_\Sigma$ ($a_i,\ a_j \in A$), that belong to some contradictory cycle and have the least weight. We proceed to step 1. The smallest weight of the arc from the asymmetric part of the relation is always greater than zero.

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PROCEDURES FOR CHOOSING BEST ALTERNATIVES BY GRAPHS

Internally stable (independent) sets

Definition 6. A subset S of vertices of a digraph without loops $G =< A, \rho >$ is called an internally stable set if for $\forall a_i \in S$: $S \cap \Gamma a_i = \emptyset$.

By Γa_i we mean the following subset of vertices: $\Gamma a_i = \{a_j \mid < a_i, a_j > \in \rho,\ a_j \in A\}$.

It follows from the definition that no two vertices of the same internally stable set cannot be connected.

In decision-making theory, and sometimes in graph theory, internally stable sets are called independent: disconnected alternative vertices are not comparable in preference.

The maximum internally stable set of a digraph is a subset of vertices that is not own subset of any other internally stable subset of this digraph.

Externally stable (dominant) sets

Definition 7. A subset T of vertices of a digraph without loops $G =< A, \rho >$ is called an externally stable set if for $\forall a_i \notin T$: $T \cap \Gamma a_i \neq \emptyset$ holds.

In decision-maker theory, externally stable sets are called dominant: the vertices-alternative from this set dominate the remaining vertices.

A minimal externally stable set of a graph is an externally stable subset such that no other externally stable subset is its own subset.

To find the maximum internally and minimal externally stable subsets by the Magu method.

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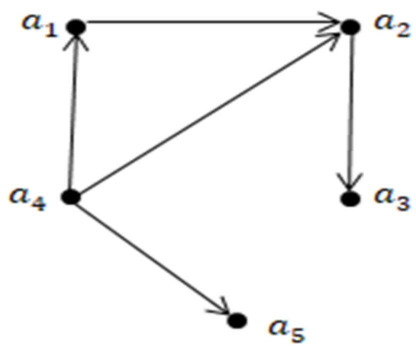
PROCEDURES FOR CHOOSING BEST ALTERNATIVES BY GRAPHS

The nucleus of a graph is a subset of vertices *N*, which is both internally and externally stable.

A nucleus of a digraph is advisable to choose as a subset of the best alternatives: these vertices-alternative are not connected with each other and dominate the rest.

Example 1.



Maximum internally stable sets (independent):

$\{a_1, a_3, a_4\}, \{a_2, a_4\}, \{a_3, a_5\}$.

Minimal externally stable sets (dominant):

$\{a_2, a_3, a_4\}, \{a_1, a_3, a_4\}$.

The nucleus of a graph: $\{a_1, a_3, a_4\}$.

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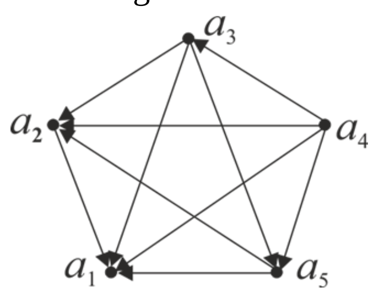
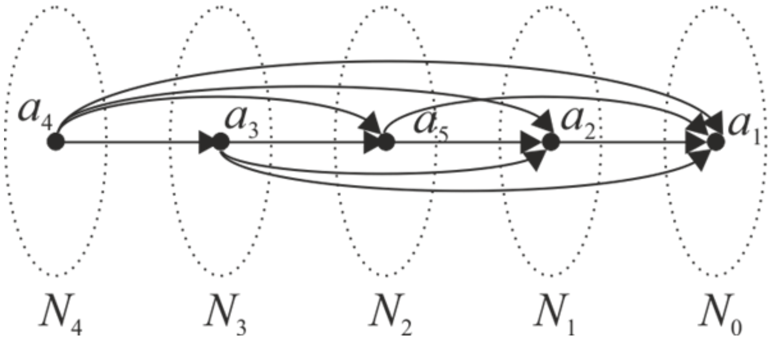
PROCEDURES FOR ORDERING ALTERNATIVES BY GRAPHS

Demukron algorithm

To order alternatives, you can use the standard procedure of graph theory, which allows you to divide the graph into levels. The disadvantage of this method is that it can be applied only if the digraph does not contain cycles.

Example 2.





$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$, $L^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \\ 2 \end{pmatrix}$. The element $l_1^{(0)} = 0$, therefore, the node $a_1 \in N_0$.

According to the algorithm, we sum the elements of each row of the vertex-adjacency matrix. We obtain the vector $L^{(0)}$.

Zero the first column of the vertex matrix (corresponding to the node a_1). And so on: $L^{(1)}, \dots, L^{(4)}$.

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Preference levels

Definition 8. The preference levels of the digraph $G =< A, \rho >$ are the sets of vertexes $P_0, P_1, \dots, P_s$ such that the vertexes of the level P_i are coincide the vertexes of components of the strong connectivity of the level $\overline{N}_i$ of the digraph $\overline{G} =< \overline{A}, \ \overline{\rho} >$ $(i = 1, \dots, s)$.

The algorithm for partitioning the digraph $G =< A, \ \rho >$ into preference levels

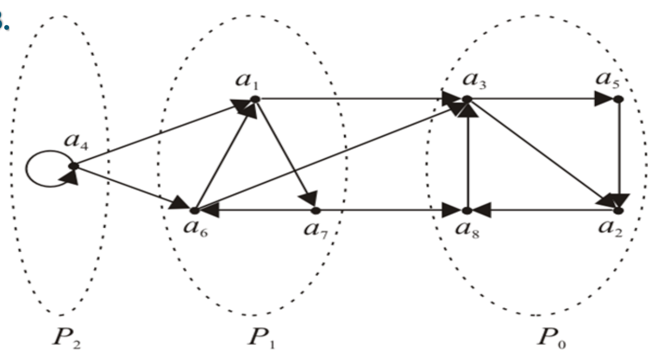
1. We find the components of the strong connectivity of the digraph $G =< A, \ \rho > \ G_1, \ G_2, \dots, G_k$

2. We construct the condensation graph $\overline{G} =< \overline{A}, \ \overline{\rho} >$ with vertexes that are components of strong connectivity $G_1, \ G_2, \dots, G_k$.

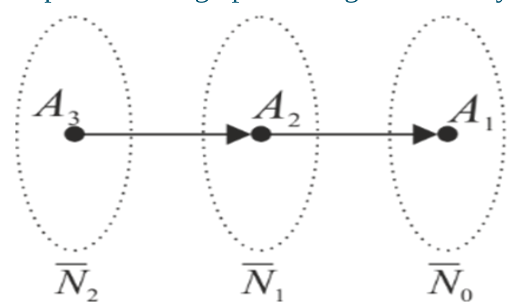
3. Using the Demukron algorithm, we partition the digraph $\overline{G} =< \overline{A}, \ \overline{\rho} >$ into the levels $\overline{N}_0, \overline{N}_1, \dots, \overline{N}_s$.

4. The vertexes of the components of the strong connectivity of the levels $\overline{N}_0, \overline{N}_1, \dots, \overline{N}_s$ of the digraph $\overline{G} =< \overline{A}, \ \overline{\rho} >$ are the vertexes of the preference levels $P_0, P_1, \dots, P_s$.

Example 3.



Ordering the components of a graph's strong connectivity.



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MATHEMATICAL MODEL OF THE SOFTWARE SYSTEM

The system is based on a mathematical model containing the following elements

$$<t, A, E, K, P, M, G, k, r>,$$

where ***t*** is the statement of the problem;

$\boldsymbol{A} = \{a_1, a_2, \ldots, a_n\}$ is the set of considered alternatives;

$\boldsymbol{K} = \{K_1, K_2, \ldots, K_m\}$ is the set of criteria;

$\boldsymbol{E} = \{E_1, E_2, \ldots, E_m\}$ is the set of experts;

P is profile of individual preferences of experts or preferences by criteria;

M is the set of preference matrices;

G is the set of digraphs of preference relationships;

$\boldsymbol{k} = \{k_1, k_2, \ldots, k_m\}$ is the set of importance coefficients of criteria and / or experts;

r is the decisive rule.

Depending on the meaningful statement of the problem *t*, it is required to rank alternatives or select the best options. Many alternatives *A* are usually asked by the decision maker (DM), but can be expanded by experts if necessary. $E_1, E_2, \ldots, E_m$. The profile *P* of individual expert preferences $E_1, E_2, \ldots, E_m$ or preferences according to the criteria $K = \{K_1, K_2, \ldots, K_m\}$ is defined on the set of alternatives *A* by binary preference relations $\rho_1, \rho_2, \ldots, \rho_m$. Preference matrices $M=\{P_1, P_2, \ldots, P_m\}$ are formed on the basis of preferences of alternatives according to criteria and / or as a result of information received from experts. The preference relations can be associated with the graphs *G* with their adjacency matrices.

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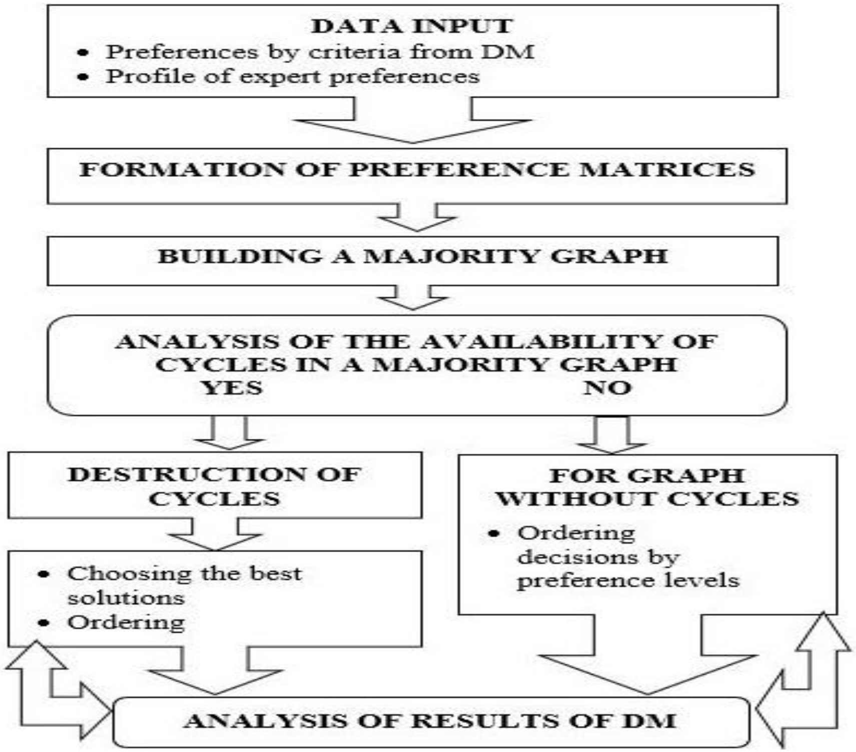
LOGIC DIAGRAM OF THE SOFTWARE SYSTEM

The main stages of the decision-making process



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Summary

A software system for supporting the decision-making process is proposed. The system is based on original algorithms for consistent aggregation of preferences and procedures for choosing and ordering alternatives by digraphs.

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Thank you for attention!

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